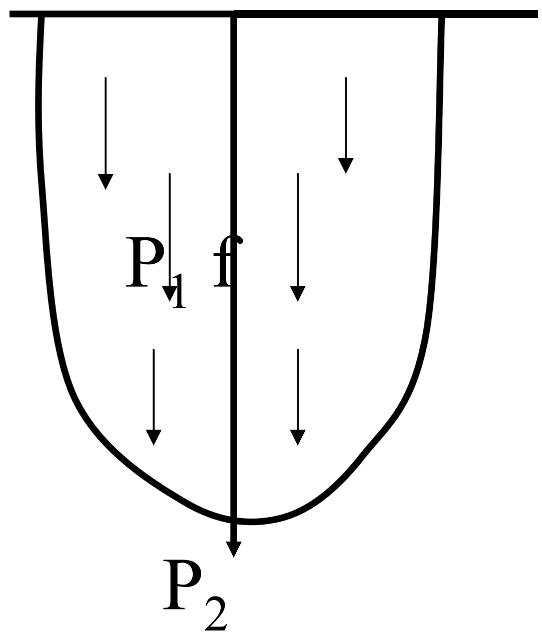
1D ELEMENTS

- Simplest type of FE problems
- All object are 1D
- All forces are 1D
- All stresses / strains are 1D

BASICS

- $u=u(x)$: Deformations
- $\varepsilon=\varepsilon(x)$: Strain
- $\sigma=\sigma(x)$: Stress
- $f=f(x)$: Body forces
- $T=T(x)$: Tractive forces
- $P=P(x)$: Point loads
- $\sigma=E\varepsilon$, $\varepsilon=du/dx$

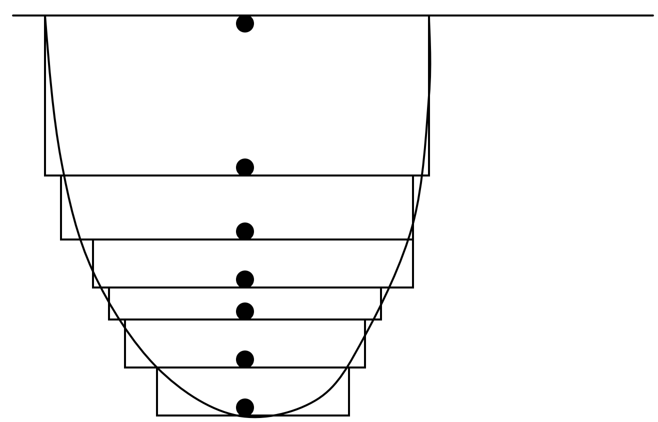
Problem



Discritization

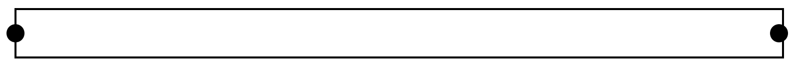
Element connectivity Matrix

Elem No.	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7



INDIVIDUAL ELEMENTS

node1 node2

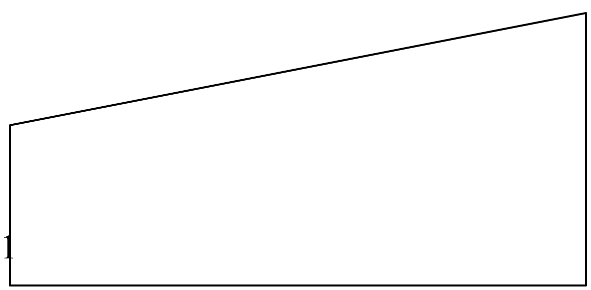


$\mathbf{x_1}$ $\mathbf{x_2}$:Coordinates

ξ=-1 ξ=1 : Local coordinates

q_1 q_2 : Deformations

(At the nodes)

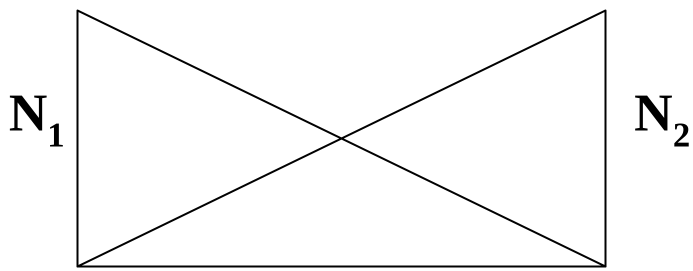


:Deformation
within the element

Linear Interpolation

N_1, N_2: Shape functions

LINEAR SHAPE FUNCTIONS



$N_1 = (1 - \xi) / 2$

$N_2 = (1 + \xi) / 2$

$$u = N_1\, q_1 + N_2\, q_2 = (1 - \xi) / 2 * q_1 + (1 + \xi) / 2 * q_2$$

Relationship between local and global coordinates-

$$\xi = -1 + 2*(x - x_1)/(x_2 - x_1)$$

STRAIN:

$\varepsilon = du/dx$

$= du/d\xi * d\xi/dx$

$u=N_1q_1+N_2q_2$

$N_1 =(1 - \xi) / 2$

$N_2 =(1 + \xi) / 2$

$d\xi/dx = 2/(x_2-x_1)$

$= 2/L_e$

$du/d\xi = (-q_1+q_2)/2$

therefore-

$\varepsilon = 1/l_e [-1 \ 1][q_1 \ q_2]^T$

$= Bq$

Where B is the element strain matrix and

$B == 1/l_e \, [-1 \;\; 1]$

As B is constant, this element is <u>CONSTANT STRAIN ELEMENT</u>

That means strain inside the element does not vary.

STRESS:

$$\sigma = EBq$$

P.E. approach

$$\Pi = \tfrac{1}{2}\int_L \sigma^T \in A dx - \int_L u^T F\, A dx - \int_L u^T T dx - \sum u_i P_i$$

$$\sum_e \tfrac{1}{2}\int_L \sigma^T \in A dx - \sum_e \int_L u^T F\, A dx - \sum_e \int_L u^T T dx - \sum Q_i P_i$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

STRAIN ENERGY FIELD	FORCED FORCE	TRACTIVE FORCE	POINT LOADS

Total energy of the body is the sum of P.E, of all Elements.

ELEMENT STRAIN ENERGY

$$U_e = \tfrac{1}{2}\int_L \sigma^T \in A dx$$

$$= \tfrac{1}{2}\int_L q^T B^T EBqAdx$$

$$= \tfrac{1}{2} EAq^T \left[\int_L B^T Bdx\right] q$$

$$= \tfrac{1}{2} EAq^T . \left[\int_L \tfrac{1}{l_e^2}\begin{bmatrix}-1 & 1\end{bmatrix}^T \begin{bmatrix}-1 & 1\end{bmatrix} dx\right] q$$

$$= \tfrac{1}{2} EAq^T \frac{1}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int dx \; q$$

$$= \tfrac{1}{2}\frac{EA}{l_e} q^T \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} q$$

$$Ue = \frac{1}{2} q^T k_e q$$

k_e *is element stiffness matrix*

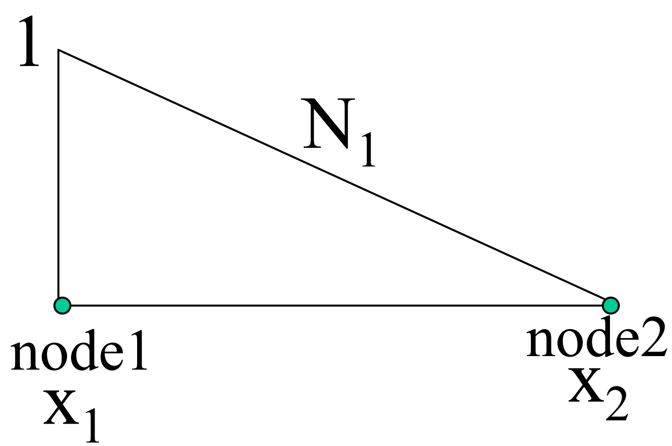
$$k_e = \frac{AE}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

CONTRIBUTION OF THE FORCES TO PE

BODY FORCES

$$\int q^T N^T fAdx = fAq^T \int N^T dx$$

$$= fAq^T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}$$



$$\int N_1 dx = area\ \ under\ \ the\ \ triangle.$$

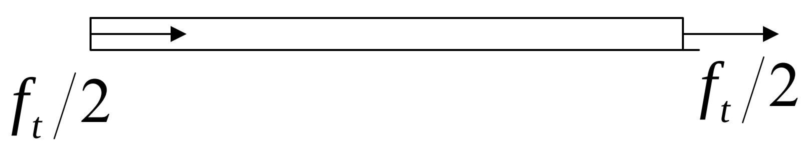
$$= (x_2 - x_1)/2 = L_e/2$$

$$= fAq^{T}\frac{l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= q^{T}\frac{fAl_e}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= q^{T}\overrightarrow{f_e} = element\ body\ force\ vector$$

Thus, body force on element gets split equally at two nodes.



TRACTIVE FORCES

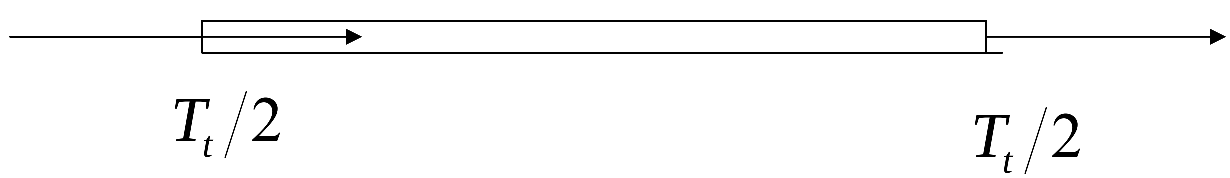
$$\int u^T T dx = \int q^T N^T T dx$$

$$= q^T T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}$$

$$= q^T \frac{Tl_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= q^T T_e$$

[TOTAL TRACTIVE FORCE ON THE ELEMENT CAN BE ASSUMED TO BE SPLIT EQUALLY AT THE NODES]



POTENTIAL ENERGY

Potential energy for an element is-

$$\Pi_e = \tfrac{1}{2} q^T k_e q - q^T f_e - q^T T_e - \sum q^T P$$

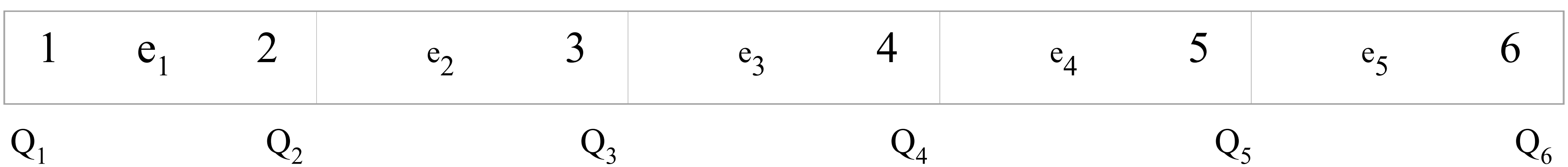
Total Potential energy-

$$\Pi = \sum \Pi_e = \sum \tfrac{1}{2} q^T k_e q - \sum q^T F_e$$

$$F_e = \left[\frac{fAl_e}{2} + \frac{Tl_e}{2} \cdot \frac{fAl_e}{2} + \frac{Tl_e}{2} \right]$$

CONSIDER A SET OF ELEMENTS

Element No.	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6



$$[q_1 \quad q_2]^T \; [q_2 \quad q_3]^T \; [q_3 \quad q_4]^T \; [q_4 \quad q_5]^T$$

ELEMENT DISPLACEMENT MATRIX

$$Q = [q_1 \quad q_2 \quad q_3 \quad - \quad - \quad - \quad q_n]^T$$

GLOBAL DISPLACEMENT MATRIX

$$\frac{E_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \frac{E_2 A_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \frac{E_3 A_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdots$$

ELEMENT STIFFNESS MATRIX

$$K=\begin{bmatrix} \frac{E_1A_1}{l_1} & -\frac{E_1A_1}{l_1} & 0 & 0 & -\ - & n \\ -\frac{E_1A_1}{l_1} & \frac{E_1A_1}{l_1}+\frac{E_2A_2}{l_2} & -\frac{E_2A_2}{l_2} & 0 & -\ - & - \\ 0 & -\frac{E_2A_2}{l_2} & \frac{E_2A_2}{l_2}+\frac{E_3A_3}{l_3} & -\frac{E_3A_3}{l_3} & & \\ & & & & & - \\ 0 & 0 & -\frac{E_3A_3}{l_3} & \frac{E_3A_3}{l_3}+\frac{E_4A_4}{l_4} & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \frac{E_nA_n}{l_n} \end{bmatrix}$$

GLOBAL STIFFNESS MATRIX

$$K = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \ldots\ldots\ldots \; n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ n \end{array} & \begin{bmatrix} k^1_{11} & k^1_{12} & 0 & 0 & 0 & 0 \ldots\ldots 0 \\ k^1_{21} & k^1_{22}+k^2_{11} & k^2_{12} & 0 & 0 & 0 \ldots\ldots 0 \\ 0 & k^2_{21} & k^2_{22}+k^3_{11} & k^3_{12} & 0 & 0 \ldots\ldots 0 \\ 0 & 0 & k^3_{21} & k^3_{22}+k^4_{11} & k^4_{12} & 0 \ldots\ldots 0 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & \ldots\ldots\ldots & & k^m_{21} & k^m_{22} \end{bmatrix} \end{array}$$

where

$k^i_{11}, k^i_{12}, k^i_{21}, k^i_{22}$: stiffness matrix elements of element number i.

PROPERTIES OF K

- Symmetric
- Banded
- Can be sparse (if numbering is not appropriate)
- Is N X N (where N is the number of nodes in a 1 D problem)

SPARSE ‘K’ MATRIX

1	6	2	3	4	5

Elem No.	Node 1	Node 2
1	1	6
2	6	2
3	2	3
4	3	4
5	4	5

THE RESULTING K MATRIX: SPARSE AND NON-BANDED

$$
K = \begin{array}{cc}
 & \begin{array}{cccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad & \quad 6 \quad \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
k^1_{11} & 0 & 0 & 0 & 0 & k^1_{12} \\
0 & k^2_{22}+k^3_{11} & k^3_{12} & 0 & 0 & k^2_{12} \\
0 & k^3_{21} & k^4_{11}+k^3_{22} & k^4_{12} & 0 & 0 \\
0 & 0 & k^4_{21} & k^4_{22}+k^5_{11} & k^5_{12} & 0 \\
0 & 0 & 0 & k^5_{21} & k^5_{22} & 0 \\
k^1_{21} & k^2_{12} & 0 & 0 & 0 & k^1_{22} + k^2_{11}
\end{bmatrix}
\end{array}
$$

ASSEMBLING GLOBAL FORCE MATRICES FROM ELEMENT MATRICES

$$\mathbf{F} = [(\mathbf{F}^e_{11} + \mathbf{P}_1)(\mathbf{F}^e_{12} + \mathbf{F}^e_{12} + \mathbf{P}_2)(\mathbf{F}^e_{22} + \mathbf{F}^e_{31} + \mathbf{P}_3)$$

$$(\mathbf{F}^e_{32} + \mathbf{F}^e_{41} + \mathbf{P}_4)\ldots]^T$$

$$= \sum (\mathbf{f}^e_i + \mathbf{T}^e_i) + \mathbf{P} = \sum \mathbf{F}^e_i + \mathbf{P}$$

Where

$\mathbf{F}^e_{i1}$: force on node 1 of element i

$\mathbf{F}^e_{i2}$: force on node 2 of element I

$\mathbf{P}_i$: point load on node number i

$$\left[\frac{fA_i l_i}{2}+\frac{Tl_i}{2} \quad \frac{fA_i l_i}{2}+\frac{Tl_i}{2}\right]$$

ELEMENT FORCE MATRIX

OR

$$\left[fi_1 \quad fi_2\right]$$

$$F=\left[f_{11} \quad f_{12}+f_{21} \quad f_{22}+f_{31} \quad f_{32}+f_{41} \quad . \quad . \quad .\right]^T$$

GLOBAL FORCE MATRIX

TOTAL PE USING GLOBAL MATRICES

$$\Pi = \sum 1/2 * q^T * k_e * q - \sum q^T * f^e - \sum q^T * T^e - \sum P_i * u_i$$

$$= \sum 1/2 * q^T * k_e * q - \sum q^T * F^e - \sum P_i * u_i$$

$$= \boxed{1/2 * Q^T * K * Q - Q^T * F}$$

where

$Q = [q_1\ q_2\ q_3\ q_4\ ....\ q_n]^T$

K : Global stiffness matrix

F : Global Force matrix

$GALERKIN'S\ \ APPROACH$

$$\int_v \sigma^T \in(\phi)\,dV - \int_v \sigma^T f\,dV - \int_s \phi T\,dS - \sum \phi^T P = 0$$

$$\phi = \phi(x)$$

$$\Rightarrow \in(\phi) = \frac{d\phi}{dx}$$

$$\phi = N\Psi$$

$$\Rightarrow \in(\phi) = B\Psi$$

$$\Psi = \begin{bmatrix}\phi_1 & \phi_2\end{bmatrix}^T$$

$$u = Nq$$

$$\in = Bq$$

$$\sigma = EBq$$

INTERNAL WORK DONE

$$\int_{\nu} \sigma^{T} \in (\phi) dV = \int q^{T} B^{T} E B \Psi A_{e} dx$$

$$= q^{T} E A_{e} B^{T} B \Psi \int dx$$

$$= q^{T} \left[E A_{e} l_{e} B^{T} B \right] \Psi$$

$$= q^{T} \left[\frac{E A_{e}}{l_{e}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \Psi$$

$$= q^{T} k_{e} \Psi$$

Virtual work due
To body forces-

$$\int_{v} \phi^{T} f dV = \int \Psi^{T} N^{T} f A_e dx$$

$$= \Psi^{T} f A_e \int N^{T} dx$$

$$= \Psi^{T} f \frac{A_e l_e}{2} \int N^{T} d\xi$$

$$= \Psi^{T} f \frac{A_e l_e}{2} \begin{bmatrix} \int N_1 d\xi \\ \int N_2 d\xi \end{bmatrix}$$

$$= \Psi^{T} f \frac{A_e l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \Psi^{T} F_e$$

$$SIMILARLY \quad TRACTIVE \quad WORK \quad DONE$$

$$\int_{v} \phi^{T} T dx = \Psi^{T} T_{e} \, where \; T_{e} = \frac{T l_{e}}{2} \begin{bmatrix} 1 & 1 \end{bmatrix}^{T}$$

Thus Galerkin's Equation Becomes

$$\int_{v} \sigma^{T} \in (\phi) dv - \int_{S} \phi^{T} f dv - \int_{S} \phi^{T} t ds - \Sigma \phi^{T} P = 0$$

$$\sum q^{T} k_{e} \Psi - \sum \Psi^{T} f_{e} - \sum \Psi^{T} T_{e} - \Psi^{T} P = 0$$

$$\Rightarrow \sum_{e} q^{T} k_{e} \Psi - \sum_{e} \Psi^{T} F_{e} = 0$$

$$\begin{bmatrix} \Psi_{1} \cdots \cdots \Psi_{i} \Psi_{i+1} \cdots \cdots \Psi_{n} \end{bmatrix}$$

$$
\begin{bmatrix}
\frac{E_1A_1}{l_1} & -\frac{E_1A_1}{l_1} & & & & \\
-\frac{E_1A_1}{l_1} & \frac{E_1A_1}{l_1}+\frac{E_2A_2}{l_2} & - & & & \\
& & & - & & \\
& & & & \frac{E_{i-1}A_{i-1}}{l_{i-1}}+\frac{E_iA_i}{l_i} & -\frac{E_iA_i}{l_i} \\
& & & & -\frac{E_iA_i}{l_i} & \frac{E_iA_i}{l_i}+\frac{E_{i+1}A_{i+1}}{l_{i+1}}
\end{bmatrix} Q
$$

$$
\begin{bmatrix}
\cdots\ \cdots\ \Psi_i \frac{E_iA_i}{l_i} - \Psi_{i+1}\frac{E_iA_i}{l_i} & -\frac{\Psi_i E_i A_i}{l_i} + \frac{\Psi_{i+1}E_iA_i}{l_i}
\end{bmatrix}
\begin{bmatrix}
Q_1 \\ . \\ . \\ Q_i \\ Q_{i+1} \\ .
\end{bmatrix}
$$

In global form-

$$\Psi^T Q\ K - \Psi^T F = 0 \ldots\ldots\ldots Galerkin's \quad approach$$

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F \ldots\ldots P.E. \quad approach$$

TO SOLVE THE PROBLEM

- **Find Deformations Q_i's**

1. **Define Boundary conditions**
2. **Apply minimization of PE**

- **Find strains $\varepsilon = B * q$**

  Find stresses $\sigma = E\, \varepsilon = E\, B\, q$

SOLVING FOR Q

$$Q = \begin{bmatrix} Q_1 & Q_2 & . & . & . & . & . & Q_n \end{bmatrix}^T$$

$$F = \begin{bmatrix} F_1 & F_2 & . & . & . & . & . & F_n \end{bmatrix}^T$$

$$K = \begin{bmatrix} K_{11} & K_{12} & . & . & K_{1n} \\ K_{21} & K_{22} & . & . & K_{2n} \\ . & & & & \\ . & & & & \\ K_{n1} & K_{n2} & & & K_{nn} \end{bmatrix}$$

$$\Pi = \frac{1}{2}\begin{pmatrix} Q_1K_{11}Q_1 + Q_1K_{12}Q_2 + Q_1K_{13}Q_3 + ...Q_1K_{1n}Q_n + \\ Q_2K_{21}Q_1 + Q_2K_{22}Q_2 + Q_2K_{23}Q_3 + ...Q_2K_{2n}Q_n + \\ . \\ . \\ Q_nK_{n1}Q_1 + Q_nK_{n2}Q_2 + Q_nK_{n3}Q_n + ...Q_nK_{nn}Q_n \end{pmatrix}$$

$$-(Q_1F_1 + Q_2F_2 + Q_3F_3 + ...Q_nF_n)$$

$$= \frac{1}{2}Q^TKQ - Q^TF$$

<u>BOUNDARY CONDITIONS</u>

$$Q_{p_1} = a_{1,} \qquad Q_{p_2} = a_2 \qquad ....$$

$$[e.g. \quad Q_2 = 0, \quad Q_5 = 0, \quad ...]$$

$$- SINGLE \quad POINT \quad CONSTRAINT$$

LET US CONSIDER AS AN EXAMPLE

$Q_1 = a_1$

$A \quad BECOMES$

$$\Pi = \tfrac{1}{2}(a_1 K_{11} a_1 + a_1 K_{12} Q_2 + a_1 K_{13} Q_3 + \ldots + a_1 K_{1n} Q_n +$$

$$+ Q_2 K_{21} a_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \ldots + Q_2 K_{2n} Q_n +$$

$$+ a_3 K_{31} a_1 + Q_3 K_{32} Q_2 +$$

.

.

$$Q_n K_{n1} a_1 + Q_n K_{n2} a_2 + \ldots\ldots + Q_n K_{nn} Q_n)$$

$$-(a_1 F_1 + Q_2 F_2 + Q_3 F_3 + \ldots + Q_n F_n)$$

$MINIMUM \quad P.E.\ PRINICIPLE \quad :$

$$\frac{\partial \Pi}{\partial Q_i} = 0$$

$$\frac{\partial \Pi}{\partial Q_2} = 0 \begin{cases} \frac{1}{2}(K_{21}a_1 + Q_2K_{22} + K_{23}Q_3 + ...K_{2n}Q_n + \\ K_{12}a_1 + Q_2K_{22} + K_{32}Q_3 + ...K_{n2}Q_n) - F_2 = 0 \end{cases}$$

$$\frac{\partial \Pi}{\partial Q_3} = 0 \begin{cases} \frac{1}{2}(K_{31}a_1 + Q_2K_{32} + K_{33}Q_3 + ...K_{3n}Q_n + \\ K_{13}a_1 + Q_2K_{23} + K_{33}Q_3 + ...K_{3n}Q_n) - F_3 = 0 \end{cases}$$

:

$$\frac{\partial \Pi}{\partial Q_i} = 0 \begin{cases} \frac{1}{2}(K_{i1}a_1 + Q_2K_{i2} + K_{i3}Q_3 + ...+ K_{in}Q_n + \\ K_{1i}a_1 + Q_2K_{2i} + K_{3i}Q_3 + ...K_{ni}Q_n) - F_i = 0 \end{cases}$$

$$Q_2K_{22} + Q_3K_{23} + ... + Q_nK_{2n} = F_2 - K_{21}a_1$$

$$Q_2K_{32} + Q_3K_{33} + ... + Q_nK_{3n} = F_3 - K_{31}a_1$$

.

$$Q_2K_{i2} + Q_3K_{i3} + ... + Q_nK_{in} = F_i - K_{i1}a_1$$

$$\begin{bmatrix} K_{22} & K_{23} & K_{24} & . & . & . & K_{2n} \\ K_{32} & K_{33} & K_{34} & . & . & . & K_{3n} \\ . & & & & & & \\ . & & & & & & \\ K_{n2} & K_{n3} & K_{n4} & . & . & . & K_{nn} \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_3 \\ . \\ . \\ Q_n \end{bmatrix} = \begin{bmatrix} F_2 - K_{21}a_1 \\ F_3 - K_{31}a_1 \\ . \\ . \\ F_n - K_{n1}a_1 \end{bmatrix}$$

$$\Rightarrow K^1Q^1 = F^1$$

K’= OBTAINED FROM K BY DELETING 1^{ST} ROW AND COLUMN

Q’= OBTAINED FROM Q BY DELETING Q_1

F’ = OBTAINED FROM F BY DELETING F_1 AND SUBTRACTING $K_{i1}a_1$ FROM F_I

IF INSTEAD OF $Q_1 = a_1$, WE HAD $Q_i = a_i$ THE SAME STEPS SHALL BE CARRIED PUT BY DOING THESE OPERATIONS ON i^{th} ROW AND COLUMN.

REACTION AT THE SUPPORT (NODE 1 IF $Q_1 = a_1$)

ELIMINATION METHOD

$$K_{11}Q_1 + K_{12}Q_2 + K_{13}Q_3 + \dots + K_{1n}Q_n = F_1 + R_1$$

$$\Rightarrow R_1 = K_{11}Q_1 + K_{12}Q_2 + K_{13}Q_3 + \dots + K_{1n}Q_n - F_1$$

ENALTY APPROACH

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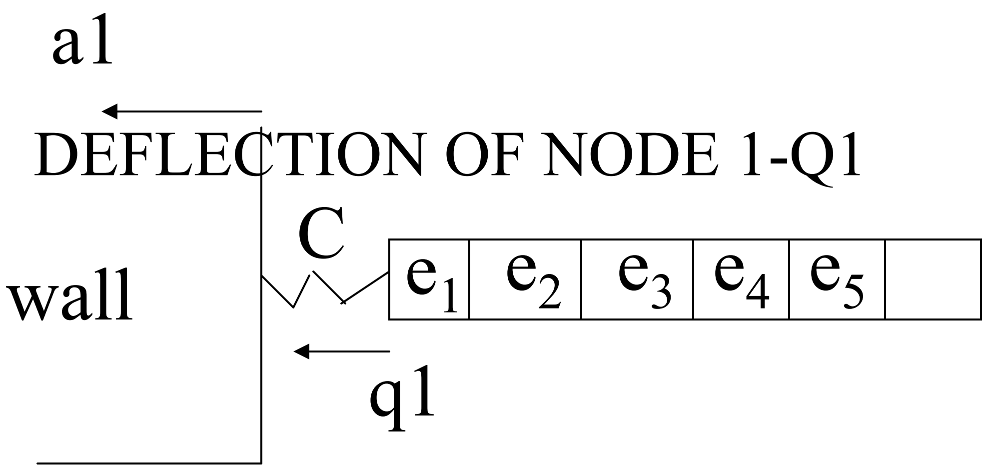
ATTACH A SPRING OF STIFFNESS C

DEFLECT THE FIXED END BY a_1

DEFLECTION OF NODE 1-Q1



AS $C \to \infty, \quad Q_1 \to a_1$

-DEFLECTION OF SPRING $\delta = Q_1 - a_1$

$$U_s = \tfrac{1}{2} C\left(Q_1 - a_1\right)^2 \left[P.E.\,of\ \ SPRING\right]$$

$$\Pi = \tfrac{1}{2} C(Q_1 - a_1)^2 + \tfrac{1}{2} Q^T KQ - Q^T F$$

$$= \tfrac{1}{2} C(Q_1 - a_1)^2 +$$

$$\tfrac{1}{2}(Q_1 K_{11} Q_1 + Q_1 K_{12} Q_2 + Q_1 K_{13} Q_3 + \ldots + Q_1 K_{1n} Q_n +$$

$$Q_2 K_{21} Q_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \ldots + Q_2 K_{2n} Q_n +$$

.

.

$$Q_2 K_{21} Q_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \ldots + Q_2 K_{2n} Q_n)$$

$$-(Q_1 F_1 + Q_2 F_2 + Q_3 F_3 + \ldots + Q_n F_n)$$

$$\frac{\partial \Pi}{\partial Q_i} = 0$$

$$\frac{\partial \Pi}{\partial Q_1} = 0 \rightarrow CQ_1 - Ca_1 + K_{11}Q_1 + K_{21}Q_2 + K_{13}Q_3 + \ldots . K_{1n}Q_n - F_1 = 0$$

$$\frac{\partial \Pi}{\partial Q_2} = 0 \rightarrow K_{21}Q_1 + Q_2K_{22} + K_{23}Q_3 + \ldots . K_{2n}Q_n - F_2 = 0$$

$$\frac{\partial \Pi}{\partial Q_3} = 0 \rightarrow K_{31}Q_1 + Q_2K_{32} + K_{33}Q_3 + \ldots . K_{3n}Q_n - F_3 = 0$$

$$\frac{\partial \Pi}{\partial Q_n} = 0 \rightarrow K_{n1}Q_1 + Q_2K_{n2} + K_{n3}Q_3 + \ldots . K_{nn}Q_n - F_n = 0$$

$$
\begin{bmatrix}
K_{11}+C & K_{12} & K_{13} & . & . & K_{13} \\
K_{21} & K_{22} & K_{23} & . & . & K_{2n} \\
K_{31} & K_{32} & K_{33} & . & . & K_{3n} \\
. & & & & & \\
. & & & & & \\
K_{n1} & K_{n2} & K_{n3} & . & . & K_{nn}
\end{bmatrix}
\begin{bmatrix}
Q_1 \\ Q_2 \\ Q_3 \\ . \\ . \\ Q_n
\end{bmatrix}
=
\begin{bmatrix}
F_1 + Ca_1 \\ F_2 \\ F_3 \\ . \\ . \\ F_n
\end{bmatrix}
$$

- LET $C = MAX(K_{11}, K_{12}, K_{13} \ldots K_{1n}) \times 10^4$
- AND SOLVE THESE EQUATIONS
- CHOOSE A LARGE VALUE OF C

Multi point constraints-

$$\beta_1 Q_{p1} + \beta_2 Q_{p2} = \beta_0$$

$$for\ example - \quad a_1 = 2a_2$$

$$we\ can\ solve\ it\ by\ penaulty\ approach$$

$$\Pi_m = \tfrac{1}{2} Q^T KQ + \tfrac{1}{2} C(\beta_1 Q_{p1} + \beta_2 Q_{p2} - \beta_0) - Q^T F$$

Since C has very large value, P.E. takes minimum value When ($\beta_1 Q_{p1}$+ $\beta_1 Q_{p1}$-β_0) is minimum.

The modified stiffness and force matrices are-

$$\begin{bmatrix} k_{p1p1} k_{p1p2} \\ k_{p2p1} k_{p2p2} \end{bmatrix} \rightarrow \begin{bmatrix} k_{p1p1} + C\beta_1^2 & k_{p1p2} + C\beta_1\beta_2 \\ k_{p2p1} + C\beta_1\beta_2 & k_{p2p2} + C\beta_2^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{p1} \\ F_{p2} \end{Bmatrix} \rightarrow \begin{Bmatrix} F_{p1} + C\beta_0\beta_1 \\ F_{p2} + C\beta_0\beta_2 \end{Bmatrix}$$

Reactions at support are given by-

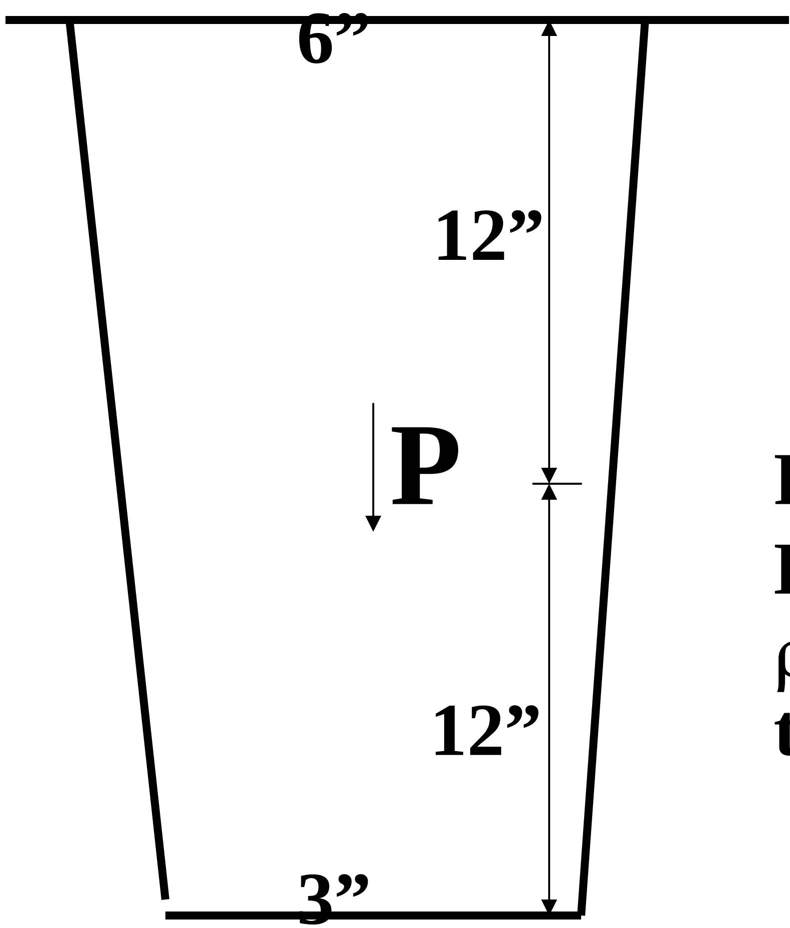
$$R_{p1} = -C\beta_1(\beta_1 Q_{p1} + \beta_2 Q_{p2} - \beta_0)$$

$$R_{p2} = -C\beta_2(\beta_1 Q_{p1} + \beta_2 Q_{p2} - \beta_0)$$

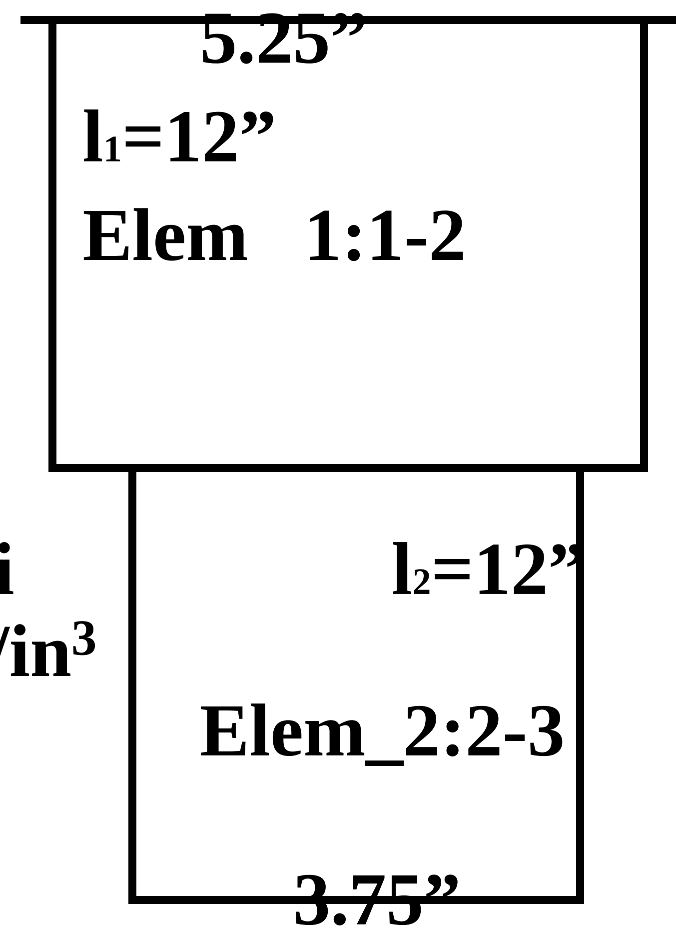
STEPS INVOLVED IN SOLVING A 1D FE PROBLEM

- Make the Geometric Model
- Make an FE Mesh
- Define the Loading
- Develop Element and Global Matrices
- Define Boundary Conditions
- Develop Modified Global Matrices
- Solving Using Numerical Techniques

A SIMPLE PROBLEM



P=100lb
E=3x10^7 psi
ρ=0.2836lb/in^3
t=1”



$$k^1 = AE/L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 15.75 * 10^7 / 12 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k^2 = AE/L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 11.25 * 10^7 / 12 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f^1 = \rho\ A\ L / 2\ [1\ 1]^T = 5.25 * 12 * 0.2836 / 2\ [1\ 1]^T$$

$$f^2 = \rho\ A\ L / 2\ [1\ 1]^T = 3.75 * 12 * 0.2836 / 2\ [1\ 1]^T$$

GLOBAL MATRICES

$$K = 3\times10^{7} / 12 \begin{bmatrix} 5.25 & -5.25 & 0 \\ -5.25 & 9.00 & -3.75 \\ 0 & -3.75 & 3.75 \end{bmatrix}$$

$$F = [8.93 \;\; 15.31+100 \;\; 6.38]^{T}$$

BOUNDARY CONDITIONS

$Q_1 = 0$

MODIFIED MATRICES

$$K' = 3\text{x}10^7 / 12 \begin{bmatrix} 5.25 & -5.25 & 0 \\ -5.25 & 9.00 & -3.75 \\ 0 & -3.75 & 3.75 \end{bmatrix}$$

$$= 3\text{x}10^7/12 \begin{bmatrix} 9 & -3.75 \\ -3.75 & 3.75 \end{bmatrix}$$

$$F' = [8.93 \ \ (15.31+100-K_{21}*a_1) \ (6.38-K_{31}*a_1)]^T$$

$$= [115.31 \ 6.38]^T$$

$$Q' = [Q2 \ Q3]^T$$

FINAL EQUATIONS

K'Q' = F'

Solving we get:

$Q' = [0.9272\ 0.9953] \times 10^{-5}$"

$Q = [\ 0\ \ 0.9272\ \ \ 0.9953] \times 10^{-5}$"

STRESSES:

$\sigma^1 = EB^1q = 3x10^7 * 1/12\ [1\ -1][0\ 0.9272x10^{-5}]^T$

$= 23.18$ psi

$\sigma^2 = EB^2q = 3x10^7*1/12\ [1\ -1][0.9273\ 0.9953]^T*10^{-5}$

$= 1.70$ psi